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COMMENT

An algorithm for characters of Hecke algebras $H_n(q)$ of type A_{n-1}

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Abstract. Recently the characters of irreducible representations of the Hecke algebra $H_n(q)$ of type A_{n-1} were identified with the transition coefficients relating q-generalized power sum symmetric functions to Schur functions (King and Wybourne 1990a, b). Using this result, we obtain an easy combinatorial algorithm for calculating the characters.

The complex Hecke algebras $H_n(q)$ of type A_{n-1} have become objects of great interest since the discovery of the Jones polynomial (Jones 1985, 1987) and the recognition of close connections between solvable problems in statistical physics and corresponding problems in the theory of knots and braids (Wadati *et al* 1989). The traces of the Hecke algebras $H_n(q)$ play an important role in applications, and they may be calculated from explicit constructions of the irreducible representations of $H_n(q)$ (Dipper and James 1987). Recently, a method was introduced in order to calculate the traces directly, without first constructing explicit representations (King and Wybourne 1990, 1991). In this letter we show that this direct method gives rise to an easy combinatorial algorithm for calculating these traces.

The Hecke algebra $H_n(q)$, with q an arbitrary but fixed complex parameter, is generated by g_i $(i = 1, 2, \ldots, n-1)$ subject to the relations

$$g_i^2 = (q-1)g_i + q \qquad \text{for } i = 1, 2, \ldots, n-1$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for } i = 1, 2, \ldots, n-2$$

$$g_i g_j = g_j g_i \qquad \text{for } |i-j| \geqslant 2.$$

It follows that for $q = 1$ the Hecke algebra is the group algebra of the symmetric group S_n, since each g_i can be identified with the transposition $s_i = (i, i+1)$.

A linear trace on the Hecke algebras was defined by Ocneanu. This theory expresses the trace of any given element of the Hecke algebra as a linear sum of traces of so-called minimal words v. Such a mininal word v has a certain connectivity class $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$, where ρ is a partition of n (King and Wybourne 1990, 1991). On the other hand, an irreducible representation π_λ of $H_n(q)$ is also characterized by a

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partition λ of n. Then it was shown by King and Wybourne (1990) that the trace of v in the irreducible representation π_λ, given by

$$\operatorname{tr} \pi_\lambda(v) = \chi_\rho^\lambda(q)$$

where $\chi_\rho^\lambda(q)$ are the Hecke algebra characters, is defined by the relation

$$p_\rho(q;t) = \sum_\lambda \chi_\rho^\lambda(q) s_\lambda(t). \tag{1}$$

Herein, $s_\lambda(t)$ is a Schur function (Macdonald 1979), and $p_\rho(q;t)$ is the q-generalized power sum symmetric function defined by King and Wybourne (1990):

$$p_\rho(q;t) = p_{\rho_1}(q;t) p_{\rho_2}(q;t) \cdots$$

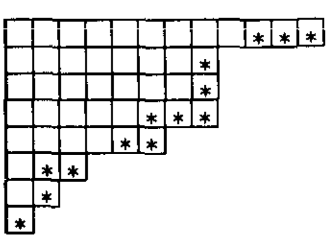
with

$$p_d(q;t) = \sum_{\substack{a,b=0 \\ a+b+1=d}}^{d-1} (-1)^b q^a s_{(a+1,1^b)}(t). \tag{2}$$

For $q = 1$, the functions $p_\rho(q;t)$ are the ordinary power sum symmetric functions, and hence it follows from (1) that in that case the Hecke algebra characters $\chi_\rho^\lambda(q)$ reduce to the characters of the symmetric group S_n (Macdonald 1979, p 62).

It follows from (1) and (2) that the Hecke algebra characters $\chi_\rho^\lambda(q)$ can be determined by calculating consecutive products of Schur functions of the type $s_{(a+1,1^b)}(t)$. This is essentially the method explained in King and Wybourne (1990). In this letter we shall show that this method can be simplified, leading to a combinatorial expression for $\chi_\rho^\lambda(q)$.

We first introduce some new objects. Let λ and μ be two partitions with $\lambda \supset \mu$ (see Macdonald (1979) for the conventions and notations concerning partitions). The skew diagram $\theta = \lambda - \mu$ is called a *boundary strip* if it contains no 2×2 block of squares. For example, if $\lambda = (12,8,8,8,6,3,2,1)$ and $\mu = (9,7,7,5,4,1,1)$ then $\theta = \lambda - \mu$ is a boundary strip, corresponding to the shaded boxes in the following Young diagram of λ:


(3)

The length of θ is equal to the number of boxes of θ and is denoted by $|\theta|$. In general θ consists of k connected parts $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(k)}$, where $\theta^{(1)}$ is the upper right section and $\theta^{(k)}$ is the lower left section. These connected parts are the *border strips* of θ (Macdonald 1979, p 31). In example (3), $\theta = \lambda - \mu$ consists of four parts or border strips. Denote by $r(\theta^{(j)})$ (respectively $c(\theta^{(j)})$) the number of rows (respectively columns) of $\theta^{(j)}$ minus 1, and let

$$r(\theta) = \sum_{j=1}^{k} r(\theta^{(j)}) \qquad c(\theta) = \sum_{j=1}^{k} c(\theta^{(j)}).$$

In example (3), we have $r(\theta^{(1)}) = 0$, $r(\theta^{(2)}) = 3$, $r(\theta^{(3)}) = 1$, $r(\theta^{(4)}) = 0$, $c(\theta^{(1)}) = 2$, $c(\theta^{(2)}) = 3$, $c(\theta^{(3)}) = 1$ and $c(\theta^{(4)}) = 0$, thus $r(\theta) = 4$ and $c(\theta) = 6$. Note that there exist some simple relations:

$$|\theta| = r(\theta) + \text{number of columns of } \theta \tag{4a}$$

$$|\theta| = c(\theta) + \text{number of rows of } \theta \tag{4b}$$

$$|\theta| = r(\theta) + c(\theta) + k. \tag{4c}$$

Definition. Let $\theta = \lambda - \mu$ be a boundary strip consisting of k connected parts. Then

$$f_\theta(q) = f_{\lambda-\mu}(q) = (-1)^{r(\theta)} q^{c(\theta)} (q-1)^{k-1}. \tag{5}$$

Lemma.

$$s_\mu(t) p_d(q;t) = \sum_\lambda f_{\lambda-\mu}(q) s_\lambda(t) \tag{6}$$

where the summation is over all λ such that $\lambda - \mu$ is a boundary strip of length d.

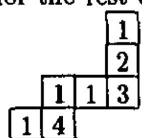
Proof. From (2) it follows that

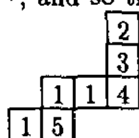
$$s_\mu(t) p_d(q;t) = \sum_{b=0}^{d-1} (-1)^b q^{d-b-1} s_\mu(t) s_{(d-b,1^b)}(t). \tag{7}$$

The idea of the proof is as follows: we use the Littlewood–Richardson rule (Macdonald 1979, p 68) to calculate $s_\mu s_{(d-b,1^b)}$, and collect the terms in the right-hand side of (7) having the same s_λ. It is quite remarkable that the q-dependent coefficient of s_λ becomes such a simple expression.

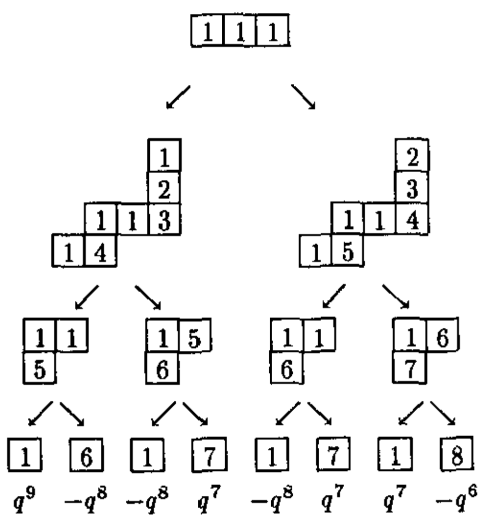
In order to calculate $s_\mu s_{(d-b,1^b)}$ one has to extend the Young diagram of μ in all possible ways to the Young diagram of a partition λ such that $\lambda - \mu$ corresponds to a tableau T consisting of $(d-b)$ 1s, one 2, one 3, ..., one b and one $b+1$, and such that $w(T)$ is a lattice permutation (Macdonald 1979, p 68). Let us first investigate whether T can have a 2×2 block of the form $\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline\end{array}$, with necessarily $\alpha \leqslant \beta$, $\gamma \leqslant \delta$, $\alpha < \gamma$ and $\beta < \delta$. β can be any of the numbers $\{1, 2, \ldots, b+1\}$, but since $w(T)$ must be a lattice permutation and every number > 1 appears only once in T, it follows that all numbers less that β have been used in previous rows and thus $\alpha = 1$. Then we must have $\gamma > \beta$ (since $\gamma > 1$ and all numbers $< \beta$ have been used), and $\delta > \beta$. But as $w(T)$ has to be a lattice permutation, we can only have $\delta = \beta + 1$. Then $\gamma > \beta$ and $\gamma \leqslant \beta + 1$ lead to only one possibility: $\gamma = \beta + 1$. This is forbidden since $\beta + 1$

can appear only once. It follows that every term s_λ in $s_\mu s_{(d-b,1^b)}$ is such that $\lambda - \mu$ is a boundary strip.

The next task is to collect all the q-dependent coefficients of a fixed s_λ in the right-hand side of (7). Before we proceed with the general proof, it is useful to consider an example. Let $\mu = (9,7,7,5,4,1,1)$, $d = 14$, and $\lambda = (12,8,8,8,6,3,2,1)$, as in figure (3). We consider all tableaux T of shape $\theta = \lambda - \mu$ of weight $(d-b, 1^b)$ $(b = 0, 1, \ldots, d-1)$ such that $w(T)$ is a lattice permutation. For the first part of θ, $\theta^{(1)}$, there is only one possible filling of the boxes, namely $\boxed{1}\boxed{1}\boxed{1}$. For the second part of θ, $\theta^{(2)}$, there are two possible choices for the number in the top right box of $\theta^{(2)}$: either 1 or 2. Once the uppermost right box of $\theta^{(2)}$ has been filled, there is no choice for the rest of $\theta^{(2)}$, and so there are two possible fillings of the boxes of $\theta^{(2)}$, namely 

and 

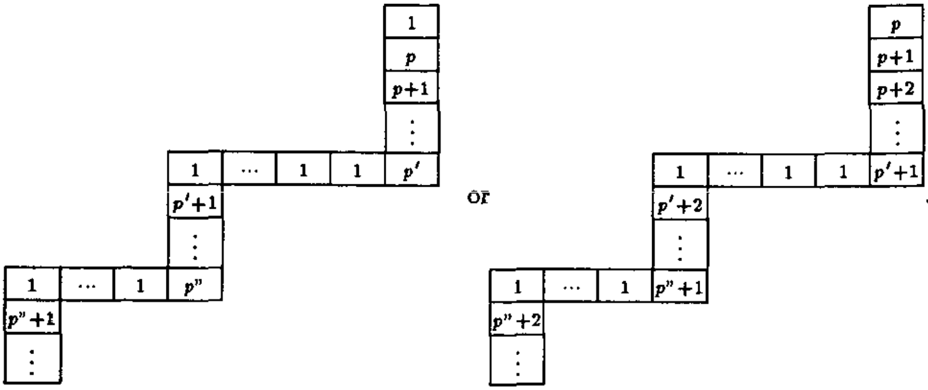
. One can proceed with $\theta^{(3)}$ and $\theta^{(4)}$, leading to the following eight possible fillings of θ:



(8)

Every path in this binary tree from the root to a leaf corresponds to an allowed tableau T of shape θ, and at each branch the numbers with which part $\theta^{(j)}$ is filled are given. For each filling one can then find the partition $(d-b, 1^b)$ from which it originates, simply by taking the largest number in the tableau and identifying it with $b-1$. Thus to every filling, and hence to every leaf of the tree, there corresponds a contribution $(-1)^b q^{d-b-1}$ from (7). These contributions have been given in (8), and one sees that their sum is equal to $q^6(q-1)^3 = (-1)^{r(\theta)} q^{c(\theta)} (q-1)^{k-1}$.

Let us now return to the proof of the general case. For $\theta^{(1)}$, the number in the uppermost right box is 1. Suppose $\theta^{(1)}, \ldots, \theta^{(j-1)}$ have been filled, and the numbers used in this filling so far are $\{1, 2, 3, \ldots, p-1\}$. Then there are only two possibilities for the number in the uppermost right box of $\theta^{(j)}$: either 1 or p. Moreover, once the uppermost right box of a connected part $\theta^{(j)}$ has been filled, there is no choice for the rest of the boxes of $\theta^{(j)}$, if we want to satisfy the lattice permutation rule. It is one of the two following fillings:



(9)

It follows that there are 2^k possible fillings of θ, with k the number of connected parts of θ, i.e. there are 2^k tableaux T of shape $\theta = \lambda - \mu$ with weight $(d-b, 1^b)$ ($b = 0, 1, \ldots, d-1$) and such that $w(T)$ is a lattice permutation. Each filling corresponds to a path from the root to a leaf in a binary tree of depth k, where every vertex at depth j is associated with a filling of $\theta^{(j)}$; in (9) the two possibilities are respectively the left and the right successor of the vertex associated with $\theta^{(j-1)}$. From (9) it follows that

for every vertex of the tree which is not a leaf, the number of 1s in the left successor is one more than the number of 1s in the right successor. (10)

To determine the q-dependent coefficient corresponding to a filling, one counts the number of 1s appearing in all the $\theta^{(j)}$ for a path in the binary tree starting at the root and ending in a leaf; this number of 1s is $d - b = |\theta| - b$, and thus the coefficient associated to the leaf is $(-1)^b q^{d-b-1}$. Consider the path to the leftmost leaf of the tree : the total number of 1s is equal to the number of columns of θ, so it follows from (4a) that the coefficient corresponding to this leaf is $(-1)^{r(\theta)} q^{|\theta| - r(\theta) - 1}$, which is equal to $(-1)^{r(\theta)} q^{c(\theta)+k-1}$ according to (4c). Consider next the path to the rightmost leaf of the tree: the total number of 1s is equal to the number of columns of θ minus $(k-1)$, so the coefficient is $(-1)^{r(\theta)+k-1} q^{c(\theta)}$. For an arbitrary leaf, let the path consist of i steps to the left and $k-1-i$ steps to the right in any order. Then it follows from (10) that the coefficient associated with this leaf is $(-1)^{r(\theta)+k-1-i} q^{c(\theta)+i}$. Therefore the sum of all coefficients at the leaves of the tree is $(-1)^{r(\theta)} q^{c(\theta)} (q-1)^{k-1}$. □

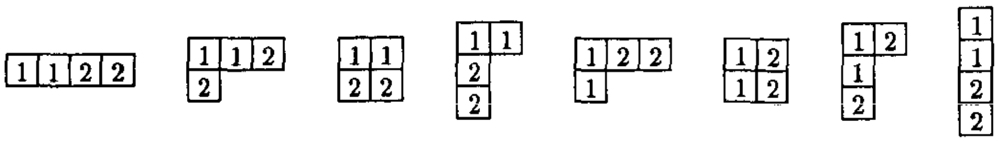
Applying the lemma for $\mu = 0$, and successively for $p_{\rho_1}(q; t)$, $p_{\rho_2}(q; t), \ldots$, and comparing with (1), one finds

Theorem. Let $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ and λ be partitions of n, then

$$\chi_\rho^\lambda(q) = \sum_S \prod_{i=1}^m f_{\lambda^{(i)} - \lambda^{(i-1)}}(q) \tag{11}$$

summed over all sequences of partitions $S = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)})$ with $0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(m)} = \lambda$ such that $\lambda^{(i)} - \lambda^{(i-1)}$ is a boundary strip of length ρ_i.

Example. Let $n = 4$ and $\rho = (2,2)$. The following gives a list of all possible diagrams built with two boundary strips of length 2. The notation is such that the boxes of $\lambda^{(i)} - \lambda^{(i-1)}$ are labelled by the number i (do not confuse this labelling with the filling of the tableaux in the proof of the lemma):



q^2 $\quad q(q-1) \quad q^2 \quad -q \quad -q \quad 1 \quad -(q-1) \quad 1$

The q-coefficient written underneath each sequence of two boundary strips of length 2 is $\prod_{i=1}^{2} f_{\lambda^{(i)}-\lambda^{(i-1)}}(q)$. Thus we obtain for $\rho = (2,2)$:

λ	$\chi_\rho^\lambda(q)$
(4)	q^2
(3,1)	$q^2 - 2q$
(2,2)	$q^2 + 1$
(2,1,1)	$-2q + 1$
(1,1,1,1)	1

The expressions for $\chi_\rho^\lambda(q)$ for special values of λ or ρ are easily obtained from (11):

$$\chi_\rho^{(n)}(q) = \prod_{i=1}^{m} q^{\rho_i - 1} = q^{|\rho| - l(\rho)} \qquad (m = l(\rho) = \text{ length of } \rho)$$

$$\chi_\rho^{(1^n)}(q) = \prod_{i=1}^{m} (-1)^{\rho_i - 1} = (-1)^{|\rho| - l(\rho)}$$

$$\chi_{(n)}^\lambda(q) = \begin{cases} (-1)^b q^{n-b-1} & \text{if } \lambda = (n-b, 1^b) \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_{(1^n)}^\lambda(q) = \chi_{(1^n)}^\lambda = \text{ number of standard Young tableaux of shape } \lambda.$$

These special cases, together with other examples, can be verified using the tables of King and Wybourne (1991).

Finally, it should be noted that a result similar to (11) has been obtained by Ram (1990), using very different techniques. The existence of this preprint was pointed out to the author after the completion of the present manuscript.

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